HOMOGENEOUS LAGRANGIAN SUBMANIFOLDS OF POSITIVE EULER CHARACTERISTIC

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ABSTRACT. We fully classify all Lagrangian submanifolds of a complex Grassmannian which are an orbit of a compact group of isometries and have positive Euler characteristic.

1. Introduction

It is natural to try to find new examples of Lagrangian submanifold of Kähler manifolds among orbits of compact Lie groups of holomorphic isometries. In [1], the authors classified all the Lagrangian submanifolds of the complex projective space which are homogeneous under the action of a compact simple Lie group G of isometries, while apparently there exist many more examples when one drops the assumption of simplicity of G; interestingly, not all the examples found in this classification have parallel second fundamental form and therefore they provide new examples. More recently, H. Ma and Y. Ohnita ([7]) classified homogeneous Lagrangian submanifolds of the complex quadrics via a neat correspondence with homogeneous isoparametric hypersurfaces of the standard sphere.

However, a full classification of homogeneous Lagrangian submanifolds (HLS) of Hermitian symmetric spaces or even of the complex projective space seems out of reach by now. In this paper we focus on a particular class of HLS of the complex Grassmannian $\operatorname{Gr}_k(\mathbb{C}^n)$, $(1 \leq k \leq [\frac{n}{2}])$ endowed with the standard Kähler structure, namely those which have positive Euler characteristic; in this case we are able to provide a full classification.

Theorem 1.1. Let G be a compact connected Lie subgroup of U(n) acting almost faithfully on the complex Grassmannian $Gr_k(\mathbb{C}^n)$ $(1 \leq k \leq [\frac{n}{2}])$ with a Lagrangian orbit \mathcal{L} with positive Euler characteristic. Then

- (a) G is simple, except for G = SO(4) and $\mathcal{L} = Gr_2(\mathbb{R}^4) \subset Gr_2(\mathbb{C}^4)$;
- (b) the orbit \mathcal{L} is the only isotropic G-orbit and it is totally geodesic; moreover the triple $(G, \mathcal{L}, \operatorname{Gr}_k(\mathbb{C}^n))$ appears in Table 1, where $\operatorname{Gr}_p(\mathbb{R}^n)$ and $\operatorname{Gr}_p(\mathbb{H}^n)$ denote the Grassmannian of real p-planes in \mathbb{R}^n and the Grassmannian of quaternionic p-planes in \mathbb{H}^n respectively.

In particular $\mathbb{R}P^{2n}$ is the only HLS with positive Euler characteristic in a complex projective space.

Conversely, every subgroup $G \subset U(n)$ appearing in Table 1 has \mathcal{L} as a Lagrangian orbit of positive Euler characteristic.

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ĺ	G	\mathcal{L}	$\mathrm{Gr}_k(\mathbb{C}^n)$
ĺ	SO(n)	$SO(n)/S(O(p) \times O(n-p)) \cong Gr_p(\mathbb{R}^n), p(n-p)$ even	$\mathrm{Gr}_p(\mathbb{C}^n)$
ĺ	Sp(m)	$\operatorname{Sp}(m)/\operatorname{Sp}(p) \times \operatorname{Sp}(m-p) \cong \operatorname{Gr}_p(\mathbb{H}^m)$	$\operatorname{Gr}_{2p}(\mathbb{C}^{2m})$
ĺ	Spin(7)	$SO(7)/U(3) \cdot \mathbb{Z}_2 \cong Gr_2(\mathbb{R}^8)$	$\mathrm{Gr}_2(\mathbb{C}^8)$
ĺ	G_2	$\operatorname{G}_2/\operatorname{U}(2)\cdot\mathbb{Z}_2\cong\operatorname{Gr}_2(\mathbb{R}^7)$	$\mathrm{Gr}_2(\mathbb{C}^7)$

Table 1.

We remark here that the uniqueness of a homogeneous Lagrangian G-orbit, when G is semisimple, is a general fact (see [1]).

In the next section, we will set up notations and give the proof of Theorem 1.1; the main tools used in the proof are the moment maps and the representation theory for compact Lie groups. We also remark that the complexified group $G^{\mathbb{C}}$ acts on the Grassmannian with an open orbit if G has a Lagrangian orbit (see the proof of Lemma 2.1), so that another approach could be to use the classification of prehomogeneous irreducible vector spaces by Sato and Kimura ([9]); we prefer to give an independent proof together with a more theoretical view of the geometric setting.

There are other situations, where a full classification of Lagrangian orbits is feasible; for instance, the following observation can be easily deduced from the classification by Brion of multiplicity free homogeneous spaces ([2])

Proposition 1.2. Let G be a compact Lie group acting isometrically and holomorphically on a Kähler manifold M in a Hamiltonian fashion. If the action of G is multiplicity free and G has a Lagrangian orbit $\mathcal L$ of positive Euler characteristic, then G is semisimple and $\mathcal L$ is locally diffeomorphic to the product of symmetric spaces of inner type.

A complete list of such actions can be easily obtained using [2].

2. Preliminaries and proof of the main theorem

Throughout the following G will denote a compact connected Lie group and $T \subset G$ a fixed maximal torus; gothic letters will indicate the corresponding Lie algebras. We also recall that a G-homogeneous space G/H has positive Euler characteristic if and only if the subgroup H has maximal rank in G (see e.g. [8]).

If G is a subgroup of $\mathrm{U}(n)$ for some $n\in\mathbb{N}$, then G acts isometrically on every complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ $(1\leq k\leq [\frac{n}{2}])$ endowed with its standard Kählerian structure; we will suppose that the action is almost faithful, i.e. the subgroup of G given by all elements of G that act trivially on the Grassmannian is a finite group. Moreover, the G-action is Hamiltonian, i.e. there exists a moment map $\mu:\mathrm{Gr}_k(\mathbb{C}^n)\to\mathfrak{g}^*$, uniquely determined up to translation, which is G-equivariant and satisfies for $p\in\mathrm{Gr}_k(\mathbb{C}^n),\,Y\in T_p\mathrm{Gr}_k(\mathbb{C}^n)$ and $X\in\mathfrak{g}$

$$\langle d\mu_p(Y), X \rangle = \omega_p(Y, \hat{X}),$$

where ω is the Kähler form and \hat{X} denotes the Killing vector field induced by X. We recall that in general an orbit \mathcal{O} is isotropic if and only if $\mu(\mathcal{O})$ is contained

in the annihilator $[\mathfrak{g},\mathfrak{g}]^o := \{ \phi \in \mathfrak{g}^* | \langle \phi, [\mathfrak{g},\mathfrak{g}] \rangle = 0 \}$, in particular if and only if $\mu(\mathcal{O}) = 0$ whenever \mathfrak{g} is semisimple (see e.g. [5]).

Now, if we embed $\operatorname{Gr}_k(\mathbb{C}^n)$ into $\mathbb{P}(\Lambda^k(\mathbb{C}^n))$ via the Plücker embedding, we may use the expression for the moment mapping of a Lie subgroup of the group of all isometries of a complex projective space (see e.g. [5]): in this way if $X \in \mathfrak{g}$ and v_1, \ldots, v_k is a basis of a k-plane $\pi \in \operatorname{Gr}_k(\mathbb{C}^n)$, then there is a moment map $\mu : \operatorname{Gr}_k(\mathbb{C}^n) \to \mathfrak{g}^*$ such that

$$\langle \mu(\pi), X \rangle = \frac{1}{||v_1 \wedge \ldots \wedge v_k||^2} (X \cdot v_1 \wedge v_2 \wedge \ldots \wedge v_k + \ldots + v_1 \wedge \ldots \wedge X \cdot v_k, \ v_1 \wedge \ldots \wedge v_k), \tag{1.1}$$

where (\cdot,\cdot) denotes the standard Hermitian product in $\Lambda^k(\mathbb{C}^n)$ induced by the standard $\mathrm{U}(n)$ -invariant Hermitian product on \mathbb{C}^n . We now suppose to have a G-orbit $\mathcal{L} = G \cdot \pi = G/H$, with $\pi \in \mathrm{Gr}_k(\mathbb{C}^n)$, of positive Euler characteristic; then we can suppose $T \subseteq H$. We can then consider the weight space decomposition $\mathbb{C}^n = \bigoplus_{\lambda \in \Lambda} V_\lambda$ w.r.t. to the torus T, where $\Lambda \subset \mathrm{Hom}(\mathfrak{t}, i\mathbb{R})$ is the set of weights; the k-plane π , being T-invariant, splits as $\pi = \bigoplus_{\lambda \in \Lambda} \pi_{\lambda}$, where $\pi_{\lambda} := \pi \cap V_{\lambda}$.

If \mathcal{L} is Lagrangian, the isotropy representation of H at π is isomorphic to $\iota^{\mathbb{C}}$, where ι is the isotropy representation $\iota: H \to O(T_{\pi}\mathcal{L})$. We now claim that

for every
$$\lambda \in \Lambda$$
, $V_{\lambda} \cap \pi \neq \{0\} \Rightarrow V_{\lambda} \subseteq \pi$.

Indeed suppose that there exists λ with $V_{\lambda} \cap \pi \neq \{0\}$ but V_{λ} not fully contained in π ; then V_{λ} intersects the orthogonal π^{\perp} non trivially. Since the tangent space $T_{\pi}\mathrm{Gr}_k(\mathbb{C}^n) \cong \mathrm{Hom}(\pi,\pi^{\perp}) \cong \pi^* \otimes \pi^{\perp}$, we see that the isotropy representation of H at π has $-\lambda + \lambda = 0$ as a weight; but the isotropy representation of H, given by $i^{\mathbb{C}}$, does not have 0 as a weight because H contains a maximal torus, a contradiction. So there is a subset $Q \subset \Lambda$ such that

$$\pi = \bigoplus_{\lambda \in Q} V_{\lambda}.$$

Moreover, since the weights of $i^{\mathbb{C}}$ are roots of $\mathfrak{g}^{\mathbb{C}}$ w.r.t. the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$, we see that the weights of $\pi^* \otimes \pi^{\perp}$ are roots, i.e.

$$\mu - \lambda \in R, \quad \forall \lambda \in Q, \ \forall \mu \in \Lambda \setminus Q$$
 (1.2)

where R is the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$.

Lemma 2.1. If G has a Lagrangian orbit \mathcal{L} in $Gr_k(\mathbb{C}^n)$ with positive Euler characteristic, then G is semisimple and it acts on \mathbb{C}^n irreducibly. Moreover the orbit \mathcal{L} is the only isotropic orbit in $Gr_k(\mathbb{C}^n)$.

Proof. (1) We prove that G is semisimple. Indeed we claim that the center Z of G is a finite group; in order to prove this, we will show that Z acts trivially on the Grassmannian and our claim will follow form the fact that the G-action is almost faithful. Now, Z acts trivially on the orbit $\mathcal{L} = G \cdot p = G/H$ because H has maximal rank and therefore contains Z; so the isotropy representation of every $z \in Z$ is trivial on the tangent space $T_p\mathcal{L}$, hence on $JT_p\mathcal{L}$, where J is the complex structure. Since $T_p\mathcal{L}$ and $JT_p\mathcal{L}$ span the whole tangent space at p, z acts as the identity.

(2) We now prove that G acts irreducibly. If not, we split $\mathbb{C}^n = W_1 \oplus W_2$ for some non-trivial G-invariant complex subspaces W_1, W_2 ; since $\Lambda = \Lambda_1 \cup \Lambda_2$ where

 Λ_i is the weight system of W_i (i=1,2), we have that $\pi=(\pi\cap W_1)\oplus(\pi\cap W_2)$. On the other hand if \mathcal{L} is Lagrangian, the complexified group $G^{\mathbb{C}}$ acts on the Grassmannian with the open orbit $G^{\mathbb{C}} \cdot \pi$: indeed, if J denotes the complex structure, $T_{\pi}(G^{\mathbb{C}} \cdot \pi) = \{\hat{X}_{\pi} + J\hat{Y}_{\pi} | X, Y \in \mathfrak{g}\} = T_{\pi}\mathcal{L} + JT_{\pi}\mathcal{L} = T_{\pi}\mathrm{Gr}_k(\mathbb{C}^n)$. But any k-plane $\pi' = u \cdot \pi$ $(u \in G^{\mathbb{C}})$ in the $G^{\mathbb{C}}$ -orbit will be of the form $\pi' = (\pi' \cap W_1) \oplus (\pi' \cap W_2)$ because $G^{\mathbb{C}}$ preserves the decomposition $\mathbb{C}^n = W_1 \oplus W_2$; since any k-plane of the form π' is not generic, the orbit $G^{\mathbb{C}} \cdot \pi$ is not open, a contradiction.

(3) Since G is semisimple, an orbit \mathcal{O} is isotropic if and only if $\mathcal{O} \subseteq \mu^{-1}(0)$; we now claim that $\mu^{-1}(0)$ coincides with the Lagrangian orbit \mathcal{L} , proving our last assertion. We first recall that the set $\mu^{-1}(0)$ is connected (see [5]), so that it is will be enough to show that \mathcal{L} is open in $\mu^{-1}(0)$. We recall that the set $G^{\mathbb{C}} \cdot \mathcal{L}$ is open in $\operatorname{Gr}_k(\mathbb{C}^n)$ because \mathcal{L} is Lagrangian; we claim that $(G^{\mathbb{C}} \cdot \mathcal{L}) \cap \mu^{-1}(0) = \mathcal{L}$, showing that \mathcal{L} is open in $\mu^{-1}(0)$. This last assertion follows from Kirwan's Lemma ([5], p. 97): if two points in $\mu^{-1}(0)$ lie in the same $G^{\mathbb{C}}$ -orbit, they belong to the same G-orbit.

Since G is semisimple, the orbit $G \cdot \pi$ is isotropic if and only if $\mu(\pi) = 0$; we claim that this condition is equivalent to

$$\sum_{\lambda \in Q} m_{\lambda} \cdot \lambda = 0, \tag{1.3}$$

where m_{λ} is the multiplicity of the weight λ , i.e. $m_{\lambda} = \dim V_{\lambda}$; indeed, if we choose a basis of π made of weight vectors, then (1.1) shows that (1.3) holds if and only if $\langle \mu(\pi), H \rangle = 0$ for every $H \in \mathfrak{t}$. A direct inspection shows that for any E in any root space $\mathfrak{g}_{\alpha} \subset \mathfrak{g}^{\mathbb{C}}$ we have that $(E \cdot \pi, \pi) = 0$ and we get our claim.

Moreover, since the *H*-representation $\pi^* \otimes \pi^{\perp}$ is equivalent to $i^{\mathbb{C}}$, that has root spaces as weight spaces, we see that dim $V_{\mu} \otimes V_{\lambda}^* = 1$ for every $\lambda \in Q$ and $\mu \in \Lambda \setminus Q$, i.e.

$$m_{\lambda} = 1, \quad \forall \ \lambda \in \Lambda.$$
 (1.4)

Remark. Using these considerations and notations, we can easily provide examples of *isotropic* orbits of positive Euler characteristic in a Grassmannian $Gr_k(\mathbb{C}^n)$. Indeed the G-orbit through the plane $\pi = \bigoplus_{w \in W} V_{w \cdot \lambda_o}$, where W is the Weyl group of G and λ_o is any weight whose W-orbit does not exhaust the set of all weights Λ is isotropic (because $\sum_{w \in W} w(\lambda_o) = 0$) and the stabilizer G_{π} has maximal rank in G; however these orbits are rarely Lagrangian.

We now consider the case k = 1, i.e. the complex projective space $\mathbb{C}P^{n-1}$; our result can be easily deduced from Lemma 2.1 and the list in [1] or the tables in [9], but we prefer to give here a more direct proof.

Lemma 2.2. If k = 1, then G is simple.

Proof. Suppose G is (locally) isomorphic to a product of simple groups $G_1 \times \ldots \times G_s$ for some $s \geq 2$ and accordingly $V = \bigotimes_{i=1}^s W_i$ for some irreducible G_i -modules W_i , $i=1,\ldots,s$; the maximal torus T splits (locally) as $T_1 \times \ldots \times T_s$ for some maximal tori $T_i \subset G_i$ and the weight system $\Lambda = \Lambda_1 + \cdots + \Lambda_s$, where Λ_i is the weight system (relative to T_i) of G_i acting on W_i , $i=1,\ldots,s$. The 1-plane π is given by some weight space $V_\lambda = \bigotimes W_{i,\lambda_i}$ with $\lambda = \lambda_1 + \ldots + \lambda_s$ for some weights $\lambda_i \in \Lambda_i$, $(i=1,\ldots,s)$; if we now select $\mu_i \in \Lambda_i$ with $\mu_i \neq \lambda_i$, we see that the weight $\mu := \sum_{i=1}^s \mu_i \neq \lambda$ and $\mu - \lambda$ is not a root, contradicting (1.2). So G is simple.

We now conclude the case k=1. By (1.3) we have that $0 \in \Lambda$ and $\pi=V_0$; moreover by (1.2) we have $\Lambda \subseteq R$. This implies that the representation is the

adjoint representation of a Lie algebra of rank 1 or \mathfrak{g} is of type \mathfrak{b}_l , \mathfrak{c}_l , \mathfrak{f}_4 , \mathfrak{g}_2 and the irreducible representation has highest weight $\lambda_1, \lambda_2, \lambda_4, \lambda_1$ respectively, where λ_i denote the fundamental weights (see e.g. [4], p. 523). Among these, only the case $\mathfrak{g} = \mathfrak{so}(2l+1)$ acting on \mathbb{C}^{2l+1} in the standard way or $\mathfrak{g} = \mathfrak{g}_2$ acting on \mathbb{C}^7 in the standard way are admissible: indeed when $\mathfrak{g} = \mathfrak{c}_l$ ($l \geq 3$) is acting on $\Lambda_o^2(\mathbb{C}^{2l})$ one easily sees that the zero weight space has dimension l-1 (or also it has no open orbits in the corresponding projective space by [9], p.105); when $\mathfrak{g} = \mathfrak{f}_4$ is acting on \mathbb{C}^{26} in the standard way, the zero weight space is two dimensional (and also the action on $\mathbb{C}P^{25}$ has no open orbits by [9] p. 143) and therefore it can be ruled out. We now deal with the case $k \geq 2$.

Lemma 2.3. If $k \geq 2$ then G is simple, unless G = SO(4) acting on $Gr_2(\mathbb{C}^4)$ with $\mathcal{L} = Gr_2(\mathbb{R}^4)$ as a Lagrangian orbit.

Proof. Suppose G is not simple and use the same notations as in Lemma 2.2, with $n_i := \dim W_i$ and $n_1 \ge n_2 \ge \ldots \ge n_s \ge 2$. First suppose $s \ge 3$: If $\lambda = \lambda_1 + \ldots + \lambda_s$ is in Q, then by (1.2), Q contains the disjoint subsets $(\Lambda_1 \setminus \{\lambda_1\}) + (\Lambda_2 \setminus \{\lambda_2\}) + (\Lambda_3 \setminus \{\lambda_3\}) + \Lambda_4 + \ldots + \Lambda_s$ and $(\Lambda_1 \setminus \{\lambda_1\}) + \lambda_2 + (\Lambda_3 \setminus \{\lambda_3\}) + \Lambda_4 + \ldots + \Lambda_s$; therefore $k \ge [(n_1 - 1)(n_2 - 1)n_3 + (n_2 - 1)(n_3 - 1) + (n_1 - 1)(n_3 - 1)]n_4 \ldots n_s$. Since we suppose $k \le \dim V/2$, we have that $n_1 n_2 n_3 \le 2(n_1 + n_2 + n_3) - 4$; from this we easily get that $n_i = 2$ for all $i = 1, \ldots, s$. Therefore $G_i \cong \mathrm{SU}(2)$ and $\dim \mathcal{L} = 2s = k(2^s - k) \ge 2(2^s - 2)$, i.e. $s \ge 2^s - 2$ and this contradicts $s \ge 3$.

When s=2, the subset Q contains $(\Lambda_1 \setminus \{\lambda_1\}) + (\Lambda_2 \setminus \{\lambda_2\})$, hence $n_1n_2/2 \ge k \ge (n_1-1)(n_2-1)+1$. This implies $n_2=2$ and $k \ge n_1$, hence $G \cong G_1 \times \mathrm{SU}(2)$, $V=W_1\otimes \mathbb{C}^2$ and $\dim \mathcal{L}=\dim G_1/H_1+2$ for some maximal rank subgroup $H_1\subset G_1$. Therefore $\dim G_1/H_1+2=k(2n_1-k)\ge n_1^2$; if the rank of G_1 is greater than one, then $\dim H_1\ge 2$ and $\dim G_1\ge n_1^2$: this never happens because G_1 is simple, hence G_1 is locally isomorphic to a subgroup of $\mathrm{SL}(n_1)$ and therefore $\dim G_1\le n_1^2-1$.

We are left with the case $\operatorname{rank}(G_1) = 1$, namely $G_1 \cong \operatorname{SU}(2)$ and $W \cong \mathbb{C}^2 \otimes \mathbb{C}^2$; this gives the Lagrangian orbit $\operatorname{Gr}_2(\mathbb{R}^4)$ inside the Grassmannian $\operatorname{Gr}_2(\mathbb{C}^4)$.

The following Lemma will be very useful (see also [1])

Lemma 2.4. Let G be a (semi)simple compact Lie group acting linearly on some complex vector space V and therefore on the Grassmannian $Gr_k(V)$ for $1 \le k \le [\frac{n}{2}]$, with $\dim V = n$. Suppose that the representation of G on V is of real (quaternionic and k is even) type. Then G has a Lagrangian orbit in $Gr_k(V)$ if and only if it acts transitively on a real (quaternionic resp.) Grassmannian, i.e on $SO(n)/SO(k) \times SO(n-k)$ ($Sp(n/2)/Sp(k/2) \times Sp((n-k)/2)$ resp.).

Proof. We denote by $\mu: \operatorname{Gr}_k(V) \to \mathfrak{g}^*$ the unique moment map for the G-action on the Grassmannian; it is well known that every isotropic G-orbit lies in the set $\mu^{-1}(0)$, which is connected (see [5]). If G has a Lagrangian orbit \mathcal{L} , then it admits no other isotropic orbit by Lemma 2.1. If the representation is of real type (quaternionic type), then $G \subset \operatorname{SO}(n)$ ($G \subset \operatorname{Sp}(n/2)$ resp.) and $\operatorname{SO}(n)$ ($\operatorname{Sp}(n/2)$ resp.) does have a Lagrangian orbit \mathcal{L} ; if G has a Lagrangian orbit \mathcal{L}' , it also has an isotropic orbit inside \mathcal{L} , hence $\mathcal{L} = \mathcal{L}'$.

Since $G^{\mathbb{C}}$ has an open orbit in the Grassmannian and the generic stabilizer has full rank, we have that

$$\dim G - \operatorname{rank}(G) \ge k(n-k) \ge 2(n-2),\tag{1.5}$$

hence

$$n \le \frac{1}{2}(\dim G - \operatorname{rank}(G) + 4). \tag{1.6}$$

Using the well known classification of irreducible representation of simple groups whose representation space V satisfies $\dim V \leq \dim G$ (see e.g. [3], p.414), we easily see that the condition (1.6) is satisfied only for the groups and representations listed in Table 2 (where we use the standard notation for fundamental weights and where we do not list dual representations of admissible ones)

g	highest weight	$\frac{1}{2}(\dim\mathfrak{g}-\mathrm{rank}(G)+4)$	$\dim V$
\mathfrak{a}_n	λ_1,λ_2	$(n^2 + n + 4)/2$	$n+1, \frac{1}{2}n(n+1)$
$\mathfrak{b}_n, \ n \geq 2$	$\lambda_1, \lambda_n (n=2,3,4)$	$n^{2} + 2$	2n+1,4,8,16
$\mathfrak{c}_n, \ n \geq 3$	λ_1	$n^{2} + 2$	2n
$\mathfrak{d}_n, \ n \geq 4$	$\lambda_1, \lambda_{n-1} (n = 4, 5, 6)$	$n^2 - n + 2$	2n, 8, 16, 32
\mathfrak{e}_6	λ_1	38	27
\mathfrak{e}_7	λ_1	65	56
\mathfrak{e}_8	_	122	_
\mathfrak{f}_4	λ_4	26	26
\mathfrak{g}_2	λ_1	8	7

Table 2.

We now study the case k=2. When \mathfrak{g} is of classical type, we will first discuss only the representations in the table different from λ_1 and we will postpone the discussion of these to the end of the proof.

If \mathfrak{g} is of type \mathfrak{a}_n , then λ_2 is not possible unless n=3: indeed the 2-plane π must be spanned by two opposite weights and this representation has never opposite weights unless n=3 and $\mathfrak{g}=\mathfrak{su}(4)\cong\mathfrak{so}(6)$ is acting on \mathbb{C}^6 .

When \mathfrak{g} is of type \mathfrak{b}_n , we have to discuss different special cases, namely: (1) n=2 and the representation is the spin representation: this is admissible, because $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$ and the spin representation goes over to the standard one; (2) n=3 and $\mathfrak{so}(7)$ acts on \mathbb{C}^8 via spin: the group $\mathrm{Spin}(7)$ acts transitively on the real Grassmannian $\mathrm{SO}(8)/\mathrm{SO}(2) \times \mathrm{SO}(6) \cong \mathrm{SO}(8)/\mathrm{U}(4) = \mathrm{SO}(7)/\mathrm{U}(3)$, hence on the Lagrangian orbit $\mathrm{SO}(8)/\mathrm{SO}(2) \times \mathrm{O}(6)$) which can be written as $\mathrm{SO}(7)/\mathrm{U}(3) \cdot \mathbb{Z}_2$; (3) n=4 and $\mathfrak{so}(9)$ acts on \mathbb{C}^{16} via spin: it is a real representation and $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$; since $\mathrm{Spin}(9)$ does not act transitively on the real two plane Grassmannian (see [8]), we rule this out by Lemma 2.4.

If \mathfrak{g} is of type \mathfrak{d}_n we just have to discuss the half-spin representations listed in the table: (1) for n=4 we have $\mathfrak{so}(8)$ acting on \mathbb{C}^8 and this is admissible; (2) for n=5 we have $\mathfrak{so}(10)$ acting on \mathbb{C}^{16} via half-spin. In either case, the weights lie in a single Weyl orbit and we can suppose that the 2-plane π contains the highest weight λ ; but $-\lambda$ is not a weight and we can rule this case out. (3) n=6 and $\mathfrak{so}(12)$ is acting on \mathbb{C}^{32} via half-spin. Again, we may suppose that π is generated by the weight spaces of λ , $-\lambda$, where λ is the highest weight; so the action has an isotropic orbit in $\operatorname{Gr}_2(\mathbb{C}^{32})$. Since \mathfrak{g}_{π} contains $\mathfrak{g}_{\lambda} := \{X \in \mathfrak{g}; X \cdot V_{\lambda} \subseteq V_{\lambda}\} \cong \mathfrak{u}(6)$ and since $\mathfrak{u}(6)$ is maximal in $\mathfrak{so}(12)$, we see that $\mathfrak{g}_{\pi} \cong \mathfrak{u}(6)$ and $\dim_{\mathbb{R}} G \cdot \pi = 30$, while $\dim_{\mathbb{C}} \operatorname{Gr}_2(\mathbb{C}^{32}) = 60$; so this case is impossible.

When $\mathfrak{g} = \mathfrak{e}_6$, the two representations λ_1 and λ_6 are dual to each other and their weight systems are $\Lambda_1 = -\Lambda_6$; moreover each of them consists of a single Weyl orbit. Therefore the 2-plane π can be supposed to contain the highest weight space (using the action of the Weyl group); on the other it should contain the opposite of the highest weight, that is not a weight.

When $\mathfrak{g} = \mathfrak{e}_7$, the irreducible representation λ_1 is of quaternionic type and has complex dimension 56; the group E_7 does not act transitively on \mathbb{HP}^{27} (see [8]), hence this case can be ruled out by Lemma 2.4.

For $\mathfrak{g} = \mathfrak{f}_4$, it is well known that 26-dimensional irreducible representation is of real type, while the group F_4 does not act transitively on any Grassmannian (see [8]); so again we rule this out by Lemma 2.4.

For $\mathfrak{g} = \mathfrak{g}_2$ acting on \mathbb{C}^7 , we recall that the group G_2 is contained in SO(7) and that it acts transitively on the real Grassmannian SO(7)/S(O(2) × O(5)) \cong $G_2/U(2) \cdot \mathbb{Z}_2$ (see e.g. [8]) which is Lagrangian in $G_2(\mathbb{C}^7)$.

We now focus on the higher dimensional Grassmannians, i.e. $k \geq 3$. The inequality (1.5) becomes

$$n \le \frac{1}{3}(\dim G - \operatorname{rank}(G) + 9) \tag{1.7}$$

and a direct inspection shows that the only possible representation in Table 2 are listed in Table 3.

\mathfrak{g}	highest weight	$\frac{1}{3}(\dim\mathfrak{g}-\mathrm{rank}(G)+9)$	$\dim V$
\mathfrak{a}_n	$\lambda_1, \lambda_2 (n \le 3)$	$(n^2 + n + 9)/3$	$n+1, \frac{1}{2}n(n+1)$
$\mathfrak{b}_n, \ n \geq 2$	$\lambda_1, \lambda_n (n=2,3)$	$(2n^2+9)/3$	2n+1,4,8
$\mathfrak{c}_n, \ n \geq 3$	λ_1	$(2n^2+9)/3$	2n
$\mathfrak{d}_n, \ n \geq 4$	$\lambda_1, \lambda_{n-1} (n = 4, 5)$	$(2n^2 - 2n + 9)/3$	2n, 8, 16
\mathfrak{e}_6	λ_1	27	27
\mathfrak{e}_7	1	45	
\mathfrak{e}_8	1	83	
\mathfrak{f}_4	1	19	
\mathfrak{g}_2	λ_1	7	7

Table 3.

Now, for $\mathfrak{g} = \mathfrak{a}_n$, we see that the second fundamental representation for $n \leq 3$ is admissible and corresponds to the dual of λ_1 for SU(3) and to λ_1 of $\mathfrak{so}(6)$ for n = 3. For $\mathfrak{g} = \mathfrak{b}_n$, λ_2 with n = 2 is the first fundamental of $\mathfrak{sp}(2)$, while we have to discuss λ_3 for n = 3; since Spin(7) \subset SO(8) and SO(8) has a Lagrangian orbit in $\operatorname{Gr}_3(\mathbb{C}^8)$ given by a real Grassmannian on which Spin(7) does not act transitively (see e.g. [8]), this case is ruled out by Lemma 2.4.

If $\mathfrak{g} = \mathfrak{d}_n$, we just have to discuss the case λ_5 for n = 5, i.e. Spin(10) acting on \mathbb{C}^{16} via half-spin. A Lagrangian orbit in $\operatorname{Gr}_3(\mathbb{C}^{16})$ would have dimension 39, while there is no subgroup of Spin(10) of maximal rank (= 5) and dimension dim Spin(10)-39 = 6: indeed such a subgroup H would contain a maximal torus T and dim H/T = 1, forcing H to be abelian, a contradiction.

If $\mathfrak{g} = \mathfrak{e}_6$ the representation space is \mathbb{C}^{27} and all the weights lie in a single Weyl orbit (see [6]); if it has a Lagrangian orbit in $Gr_3(\mathbb{C}^{27})$, we can suppose that the

3-plane $\pi \in \operatorname{Gr}_3(\mathbb{C}^{27})$ contains the highest weight space. Now, π^{\perp} is the sum of 24 one-dimensional weight spaces V_{μ} , where $\mu = \lambda_1 - \alpha_{\mu}$, α_{μ} being a positive root with $\langle \lambda_1, \alpha_{\mu} \rangle \neq 0$. So there would be at least 24 positive roots whose expression in terms of simple roots $\{\alpha_1, \ldots, \alpha_6\}$ have a non zero coefficient relative to α_1 ; but there are only 20 such roots in \mathfrak{e}_6 (see e.g. [4]).

The case \mathfrak{g}_2 can be also ruled out using Lemma 2.4 and the fact that G_2 does not act transitively on $Gr_3(\mathbb{R}^7)$.

Using Table 2 it is immediate to see that when $k \geq 4$, there are no other admissible representation but the first fundamental ones.

We now have to discuss the first fundamental representations; for \mathfrak{b}_n and \mathfrak{d}_n , we know that the real Grassmannians are Lagrangian orbits and therefore there are no others; we just have to discuss the case of $\mathfrak{g} = \mathfrak{c}_n$ acting on a Grassmannian $\operatorname{Gr}_{2k+1}(\mathbb{C}^{2n})$ for some $1 \leq k \leq n-1$. In this case, there is no Lagrangian orbit: indeed, suppose the orbit through $\pi \in \operatorname{Gr}_{2k+1}(\mathbb{C}^{2n})$ is Lagrangian; now, π would be spanned by an odd number of weight spaces relative to some weights $\mu_1 \dots \mu_{2k+1}$ with $\sum_{i=1}^{2k+1} \mu_i = 0$. But the root system R of \mathfrak{c}_n is given by $R = \{\pm 2e_i \ (1 \leq i \leq n), \ \pm e_i \pm e_i \ (1 \leq i \neq j \leq n)\}$ (see e.g. [4] p. 463) and the weight system is simply $\Lambda = \{\pm e_i, \ 1 \leq i \leq n\}$; so it is immediate to see that the condition $\sum_{i=1}^{2k+1} \mu_i = 0$ cannot happen.

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